Numerical modeling of magnetohydrodynamic non-Newtonian flow in a cross-slot

K. Periyadurai 1,2 | J. C. Pascoa 1 | M. Abdollahzadehsangroudi 1 | P. J. Oliveira 1

1Department of Electromechanical Engineering, C-MAST - Center for Mechanical and Aerospace Science and Technology, Universidade da Beira Interior, Covilhã, Portugal

2Department of Mathematics, School of Arts, Sciences, Humanities & Education (SASHE), SASTRA Deemed-to-be-University, Thanjavur, India

Abstract
This study numerically investigates the characteristics of a non-Newtonian magnetohydrodynamic flow in a cross-slot. Numerical simulations are performed using power law, Bird–Carreau and Casson non-Newtonian fluid models. The flow characteristics and shear viscosity behavior in the flow region are analyzed for different values of magnetic field. Additionally, the dynamic behavior of the bifurcated flow in the cross-slot is studied in detail, and the computational results are compared with existing models. It is shown that the fluid velocity is reduced in both the inlet and outlet channels of the cross-slot due to the magnetic field, regardless of the viscosity model. Moreover, it is found that the fluid viscosity increases along the centerline of the inlet channels and decreases in the outlet channels of the cross-slot for all non-Newtonian fluid models tested here. Furthermore, this study shows that the flow properties of the non-Newtonian fluids can be controlled by changing the magnetic field strength. The results of this study will be useful for analyzing the flow behavior of blood in a microfluidic cross-slot and other rheological fluids used in biochemical engineering and industrial processes, where higher mass transfer and mixing efficiency can be achieved by imposing an external magnetic field.

Highlights
• Simulation of non-Newtonian magnetohydrodynamic flow in a cross-slot is conducted.
• Analysis of the flow and shear viscosity behavior for different Hartman number is performed.
• Flow properties were successfully controlled with the imposed external magnetic field.

Keywords
cross-slot, magneto-hydrodynamics, non-Newtonian fluids, numerical simulation

Abbreviations: GNF, generalized non-Newtonian fluid; MHD, magnetohydrodynamics.
INTRODUCTION

Flows in bifurcations (such as T-devices or the cross-slot) occur frequently and are found in a variety of applications, including biochemistry, drug delivery, medical diagnostics, chemical synthesis, and micro-mixing or micro-rheology. Therefore, studies of fully developed, laminar, axial flows of non-Newtonian fluids through channels, such as wide plane slots, circular pipes and centered annuli of arbitrary aspect ratio, are often needed for the design of devices for the above applications and they may also be important as reference cases for the study of more complex flows. Recent developments in numerical methods for simulating non-Newtonian fluid flows currently used in computational rheology to evaluate the performance of numerical methods and other open issues in numerical methods, such as novel and challenging applications of non-Newtonian fluids, some of which require further developments in numerical methods, are discussed by Alves et al. Generalized Newtonian fluid (GNF) models which assume that the viscosity of a non-Newtonian fluid can be modeled inelastically as a time-independent, that is, instantaneous, scalar function of the rate of strain, are used frequently. Classical examples of such models include the power law, Carreau and Cross models, among many others. Although these models have not (yet) found widespread adoption within the literature, the benefits and utility of such models in general engineering applications remain obvious.

Poole et al. investigated the effect of shear-thinning on the power law model to analyze the characteristics of non-Newtonian fluid through the critical Reynolds number. In particular, they used the power law model to investigate the effect of shear-thinning in the power law index range 0.5–1. They found that it would be possible to state that shear-thinning either stimulates or restricts the bifurcation to asymmetric flow. Matos and Oliveira have conducted a numerical study on two-dimensional bifurcation flow with non-Newtonian fluids, whose characteristics for the base case are similar to blood. Their results showed that the GNF fluids tend to have longer recirculation lengths in the main and the side branches and reduced shear stress field magnitudes when compared to corresponding Newtonian flows for the same value of inertia. Also, for unsteady flows, the behavior is almost linear and monotonous, with decreasing recirculation lengths and increasing shear stress magnitudes as the power law exponent is increased. The effects of Reynolds number and power law index on the non-Newtonian inelastic flows through a T-channel have been investigated by Khandelwal et al. They showed that the critical Reynolds number at which the onset of flow separation takes place in the side branch decreases with the decrease in the power law index. The maximum values of viscosities appeared at the bottom wall of the side branch channels for different values of Reynolds numbers and power law indices. Numerical simulation of a non-Newtonian fluid in a T-shape microchannel has been analyzed by Zimmerman et al. They noticed that by varying the channel size, it could be possible to control the flow phenomenon for the Carreau fluid even for a much larger range of time-shear relaxation parameter values. The simulation of bifurcation of blood flow with three different constitutive models has been analyzed by Abhuggattas et al. Their results showed that power law fluid has a lower wall shear stresses, especially in near the walls as compared to those predicted by using Cross and Carreau–Yasuda models.

Several aspects of the flow of a non-Newtonian fluid in a cross-slot have been analyzed in the literature. In an early publication by Scrivener et al., studies of dynamic drag reduction in a cross-slot flow were performed. They showed the dynamic behavior of the molecules and the flow field using both experimental and numerical approaches. Their findings suggested a strong relation between the elastic properties of the macromolecules and the elongational viscosity with drag reduction. Kalashnikov and Tsiklauri studied the effect of polymer additives on planar flow in a cross-slot using a three-dimensional model. They found that as the inertial forces become large with increasing Reynolds number, the viscous force is unable to decelerate the flow in the central part of the cross-slot. In addition, they found that the elastic properties of the fluid do not significantly alter the flow behavior. Schoonen et al. also explored the cross-slot geometry and performed a numerical and experimental investigation with viscoelastic fluids in a three-dimensional flow model. They described the stagnation flow in the interaction zone of the cross-slot, which exhibits strong elongational deformations. In a stagnation flow, viscoelastic models produced the lowest stresses near the stagnation line. The authors proposed the model of Phan-Thien and Tanner (PTT) to improve the stress prediction near the elongational flow zone. Peters et al. evaluated the flow of a polymer melt in a cross-slot using extended constitutive models based on the Giesekus and PTT models. They found that steady-state conditions are achieved in a region surrounding the stagnation point in the cross-slot flow. Moreover, they also showed that the extended models have the ability to predict the flow properties more accurately compared to the regular PTT and Giesekus models. The planar elongational flow in a cross-slot with inertial effects on polymer chain scission was studied by Islam et al. They compared the results of contracting flows and stagnation point elongational flows at different Reynolds numbers, suggesting that the polymer...
mass distribution in a planar elongational flow is significantly affected by the inertial properties of the flow with respect to the scission mechanism.

In the previous studies, the flow in the cross-slot was predicted to be symmetric about the two coordinate axes. Poole et al. performed numerical calculations to simulate viscoelastic fluid flow through a 2D cross-slot geometry at low Reynolds numbers, confirming the experimental results of Arratia et al. A definitive elastic instability leads to an asymmetric, but steady flow pattern when the elasticity parameters (Weiszigen, Wi and Deborah, De numbers) surpass a critical value. At high values of Wi and De, the flow remains asymmetric but becomes unsteady. Cruz et al. analyzed stationary bifurcation solutions occurring in two-dimensional cross-slot geometries under inertia-free conditions, using different constitutive models. Their goal was to provide benchmark solutions for the non-Newtonian constitutive models employed. A viscoelastic cross-slot flow was also considered by Chaffin and Rees, who demonstrated an explicit solution at the stagnation point for inlet and outlet channels of different widths. Rostami and Morini analyzed Newtonian and non-Newtonian droplet flow in a micro cross-junction. Experimental results indicated that the value of the volumetric flow rate ratio, where droplets form in the center of cross-junction, is affected by the viscosity ratio. Finally, they found that the degree of uniformity of the droplets produced by the cross-junction depends on the capillary number and the given flow rate ratio.

Researchers have paid considerable attention to the study of magnetohydrodynamic (MHD) flow problems associated with non-Newtonian fluids due to their diverse applications in engineering and industrial manufacturing. Examples include MHD power generators, fusion of metals by applying a magnetic field in an electric furnace, cooling nuclear reactors, plasma studies, and using nonmetallic inclusions to purify molten metals. The flow of incompressible non-Newtonian fluids between parallel plates in the presence of a magnetic field is described in an early work by Sarpkaya. They have compared the results for velocity and flow rate at different Hartmann numbers with those for Newtonian fluids. They proposed analytical results for the planar (2D) channel and used to approximate the flow behavior in a rectangular channel. Cobble studied the MHD flow for a power law fluid over a flat plate and under a pressure gradient. The author provided results for several non-Newtonian models including Newtonian fluid. His numerical solutions were presented using similarity variables, and he concluded that the velocity profiles are flattened by the magnetic field. Andersson et al. studied the MHD flow behavior for an electrically conducting power law fluid on a stretching sheet. They investigated how the flow characteristics are affected by the magnetic field and found that an increase in wall friction correlates with a decrease in boundary layer thickness in the channel. The effects of electromagnetic fields in parallel and cross-field systems were analyzed by Koyama et al. based on different rheological models. They found that the parallel field gives a remarkable electromagnetic effect in the suspension of the fluids, while the magnetic effect in the crossed field is not significant due to the increase in stress value. Yang et al. studied the MHD flow with natural convection of non-Newtonian fluids on a wavy vertical wall. The effect of Lorentz force on the flow properties of non-Newtonian fluids was numerically analyzed. They found that the magnetic field reduces the velocity profile and heat transfer rate due to the damping effect.

Haik et al. experimentally and numerically investigated the effect of a magnetic field on the apparent viscosity of human blood. They concluded that the applied magnetic field leads to an increase in the viscosity of the blood flow, thus decreasing the flow velocity. The analytical solution for MHD non-Newtonian fluid flow over a stretching sheet was analyzed by Liao. This solution for the skin friction coefficient on the stretching sheet can be used in many industrial applications. Physically, the author pointed out that the skin friction is enhanced by the effect of magnetic field, particularly for shear thickening fluids compared to shear thinning fluids. Xu and Liao studied the unsteady MHD flow of a non-Newtonian fluid within a stretching plate and investigated the effect of the power law index in the non-Newtonian model. They showed that the magnetic field increases the wall friction for larger power law index numbers. Hayat et al. studied the peristaltic flow of a non-Newtonian fluid under a magnetic field in a channel with a long wavelength and a low Reynolds number. Analytical solutions were obtained for the axial velocity and pressure gradient in the form of stream function equations. The frictional force and pressure rise are more pronounced for an MHD fluid than for a purely hydrodynamic fluid. The axial velocity decreases with higher values of the Hartmann number.

Kajneres reports a comprehensive mathematical study of blood flow under nonuniform magnetic fields. The time-dependent wall shear stress for various stenosis growth rates and the effect of an imposed nonuniform magnetic field on blood flow patterns are analyzed. He found that a nonuniform magnetic field induces significant changes in the flow field, making this technique applicable for optimizing targeted drug delivery. Computational modeling of blood flow as a non-Newtonian fluid in a stenosed artery with a static magnetic field was performed by Alshare et al. These authors compared the obtained computational results with previous experimental models and found that the pressure drop and wall shear stress increase with an increase in the magnetic
field for the non-Newtonian fluid. Shahidian et al. \(^{31}\) performed an analysis of non-Newtonian blood flow in the presence of a magnetic field. They observed that an increase in the applied magnetic field causes higher fluid velocity, resulting in a corresponding increase in the volumetric flow rate of the pump. Both effects lead to an increase in the magnetic flux density. Nadeem and Akbar \(^{32}\) presented the effect of an inclined magnetic field on the non-Newtonian flow of a Jeffery model in coaxial pipes. They observed that the flow velocity decreases with the inclination angle, while it increases with the magnetic field. Akbar and Khan \(^{33}\) studied the effect of a magnetic field on the non-Newtonian flow of a Jeffery model in coaxial pipes. They observed that the flow velocity decreases near the channel walls due to an increase in Hartmann number. Rehman et al. \(^{34}\) reported a comparative study of MHD boundary layer flow with mixed convection of a Casson fluid with flat and cylindrical stretching surfaces. The dimensionless velocity profiles were presented for different physical parameters, concluding that the velocity profile increases for higher values of mixed convection, while showing opposite behavior under the influence of magnetic field and Casson fluid parameter. Pourjafar et al. \(^{35}\) performed a theoretical study of MHD flow with Bingham fluid for a plane channel flow. Their results indicate that the flow kinematics under creeping flow conditions are significantly affected by the yield stress. In addition, the yield stress may have a stabilizing effect on the magnetic fluid flow due to the increased shear stress on the wall with the application of a magnetic field.

To the best of our knowledge and based on the literature review, it appears that no study has been conducted to investigate the effect of magnetic field on non-Newtonian flow in a cross-slot. Due to its practical interest for various applications mentioned above, the subject requires further research. Therefore, we propose a study to investigate the flow characteristics and the shear viscosity behavior of non-Newtonian inelastic fluids in a cross-slot under the influence of a magnetic field. The results will contribute to a better understanding of the non-Newtonian flow behavior in a cross-slot.

2 MATHEMATICAL FORMULATION

2.1 Flow governing equations and characteristic variables

We consider a two-dimensional cross-slot with a pair of inlet and outlet openings of equal length (l) and width \((D)\), as illustrated in Figure 1. Cartesian coordinates \((x,y)\) with corresponding velocity components \((u,v)\) are chosen. In this 2D schematic representation of the cross-slot geometry, the coordinate system is centered at the midpoint of the cross-slot, serving as the reference point. A magnetic field \(B_0\) is applied to the cross-slot perpendicular to the x-axis, as shown in Figure 1. The flow is assumed to be incompressible and isothermal. Under these assumptions, the governing equations for the unsteady MHD flow of non-Newtonian fluids are formulated as follows:

\[
\nabla \cdot \mathbf{u} = 0, \quad (1)
\]

\[
\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nabla \cdot \mathbf{\tau} + \mathbf{j} \times \mathbf{B}. \quad (2)
\]

We define the magnetic Reynolds number as \(Re_m = \frac{\mu m \sigma u_0 D}{\mu m + \sigma}\), where \(\mu m\) and \(\sigma\) represent the magnetic permeability and electrical conductivity of the fluid, respectively, and \(u_0\) is the imposed average velocity in the inlet channel. In this context, \(Re_m\) characterizes the ratio between momentum advection and magnetic diffusivity. In this study, we maintain a fixed magnetic Reynolds number to ensure it remains much smaller than unity. When \(Re_m \ll 1\), the induced magnetic field is negligible compared to the external applied magnetic field and we can assume \(\mathbf{B} \approx \mathbf{B}_0\). In this case, the coupling between the magnetic field and the fluid velocity is greatly weakened due to the lesser influence of the self-induced magnetic field. Consequently, the magnetic field is determined only by the boundary conditions specifying the orientation and

![Figure 1](https://4spepublications.onlinelibrary.wiley.com/doi/10.1002/pen.26683)

**Figure 1** Schematic of the physical model.
magnitude of the applied field \( B_0 \). Under this assumption, the Lorentz force is determined by Ohm’s law for current density and an electrical conservation equation,

\[ j = \sigma (-\nabla \phi + u \times B_0), \tag{3} \]

written in terms of the electric potential \( \phi \) (\( E = -\nabla \phi \)). In the above equation, the electrical conductivity of fluid \( (\sigma) \) is assumed to be constant under the conditions of the present study. The conservation of charge is:

\[ \nabla \cdot j = 0. \tag{4} \]

If we combine Equations (3) and (4), we get a Poisson equation for the electric potential:

\[ \nabla^2 \phi = \nabla \cdot (u \times B_0), \tag{5} \]

from which the current density \( j \) needed in the momentum equation can be obtained using Equation (3). In the above equations, \( u = (u, v) \) is the velocity vector, \( \tau \) is the stress tensor, \( p, \rho, \) and \( t \) are the pressure, density, and time, respectively.

The non-dimensionalization of a non-Newtonian flow problem requires the selection of a characteristic viscosity. A characteristic deformation rate \( \dot{\gamma}_c \) is defined as:

\[ \dot{\gamma}_c = 3 \left( \frac{u_0}{D/2} \right), \tag{6} \]

where \( u_0 \) is the average velocity in each arm. The viscosity varies with shear rate \( \dot{\gamma} \) as prescribed by a viscosity function \( \eta(\dot{\gamma}) \) specific to each rheological model, and a characteristic viscosity value is then given by:

\[ \eta_c = \eta(\dot{\gamma}_c). \tag{7} \]

The governing non-dimensional parameters for this flow problem are based on characteristic quantities and they include the Reynolds number \( Re = \rho u_0 D/\eta_c \) and the Hartmann number \( Ha = B_0 D/\sqrt{\sigma/\eta_c} \). The characteristic velocity \( (u_0) \) is calculated for each non-Newtonian model, considering the prescribed Reynolds number and accounting for the characteristic viscosity according to Equations (6) and (7).

In this study, the flow is considered isothermal, eliminating the need for an equation for the conservation of energy. The constitutive equations required to obtain the stress tensor in terms of fluid deformation are discussed below. As the system of conservation equations has more unknowns than equations, an additional equation is needed to close the system. Such a constitutive equation for a structurally simple Newtonian fluid is given by

\[ \tau = 2\eta D - \frac{2}{3} \gamma (\nabla \cdot u) I, \tag{8} \]

and in the isothermal and incompressible case, it becomes

\[ \tau = 2\eta D, \tag{9} \]

where \( \eta \) is the constant dynamic viscosity and \( D \) is the rate of strain tensor, defined as

\[ D = \frac{1}{2} (\nabla u + \nabla u^T). \tag{10} \]

For a non-Newtonian fluid, the local shear rates and the local shear stresses in the fluid have a nonlinear relation, which means that a constant of proportionality cannot be defined. Therefore, viscosity is a variable and not a fixed scalar. In addition, viscosity depends on either the shear rate alone or the whole shear rate time history. If the primary rheological characteristic of the fluid is a time-independent, shear rate-dependent viscosity, the fluid can be modeled using the generalized Newtonian constitutive equation,

\[ \tau = 2\eta(\dot{\gamma}) D \quad \text{where} \quad \dot{\gamma} = \sqrt{2(D : D)}, \tag{11} \]

where the viscosity model is specified by the function \( \eta(\dot{\gamma}) \). From Equation (11), \( \dot{\gamma} \) is twice the magnitude of the tensor \( D \) and in a simple shear flow, it becomes equal to the shear rate. Fluids with shear-rate-dependent viscosity \( \eta(\dot{\gamma}) \) can be classified into three categories: shear-thinning, shear-thickening, and yield-stress models. We have chosen three types of non-Newtonian fluids for the current study:

i. **Power law model**

This power law fluid is the simplest type of generalized non-Newtonian fluid (GNF) model. It provides a simple relationship for stress as a function of strain rate raised to a certain power. The relation for viscosity is then given by,

\[ \eta = k \dot{\gamma}^{n-1}, \tag{12} \]

where \( k \) is the flow consistency index and \( n \) is the power law index. Specific values of these parameters proposed by Reference 36 for blood flow are given in Table 1. Although hemodynamics is not the main focus of the present study and that all our results, in fact, will be given dimensionless, we have chosen the non-Newtonian and other physical properties as pertaining to blood...
because it is a well characterized fluid and many studies
mentioned in the introduction were related to blood flow.
This power law model does not accurately predict the
behavior and viscosity of fluids at very high shear rates
for the shear thickening fluids \((n > 1)\) or very low shear
rates for shear thinning fluid \((n < 1)\): at these extremes,
the viscosity estimated by the power law model tends to
infinity instead of reaching a constant value, as observed
experimentally.\(^{38}\) Despite this limitation, the power law
model is still widely used because of its simplicity, espe-
cially in analytical studies. In numerical simulation,
artificial limits for the viscosity values, \(\eta_{\text{min}} = 0.00345 \text{ Pa.s}\) and \(\eta_{\text{max}} = 0.056 \text{ Pa.s}\), are set to avoid
numerical convergence issues.

### ii. Bird–Carreau model
This model is valid over the entire range of shear rates.
This can have a significant effect on shear viscosity
compared to the power law model, so it is important
to incorporate the values of viscosity at zero and infi-
nite shear rates in the formulation. In this model,
viscosity is related to shear rate by the following equation:
\[
\eta = \eta_\infty + (\eta_0 - \eta_\infty) \left[1 + (\dot{\gamma} \lambda)^n \right]^{-\frac{1}{n}},
\]
where \(\eta_0\) is the viscosity at zero shear rate \((\dot{\gamma} \rightarrow 0)\), \(\eta_\infty\) is
the viscosity at infinite shear rate \((\dot{\gamma} \rightarrow \infty)\), \(\lambda\) is often
designated the relaxation time although its actual
meaning is the inverse of shear rate beyond which the
viscosity starts decreasing, and \(n\) is again a power law
index (rate of decrease of \(\frac{\eta_0 - \eta_\infty}{\eta_0 - \eta}\) in log–log scales). The
values of the parameters Bird–Carreau model are adopted
from Reference \(37\) for blood flow and are given in
Table 1.

### iii. Casson model
The Casson model is a basic model for describing the
behavior of fluids that exhibit a certain degree of
yield stress, such as ink. The constitutive relationship
for the Casson model in terms of stress is given by
\[
\sqrt{\tau} = \sqrt{\tau_0 + \sqrt{\mu \dot{\gamma}}} \quad \text{for} \quad |\dot{\gamma}| > \tau_0,
\]
and \(\dot{\gamma} = 0\) for \(|\dot{\gamma}| \leq \tau_0\).

In this constitutive model, the material is assumed to
be rigid when the local shear stress magnitude is less
than the yield stress, or \(|\dot{\gamma}| \leq \tau_0\). The material will flow
only if \(|\dot{\gamma}| > \tau_0\) or post-yield condition. That is, when the
shear stress \(\tau\) falls below \(\tau_0\), a solid-like structure is
formed (unyielded). In the above, \(|\dot{\gamma}|\) is the magnitude of
the stress tensor given by
\[
|\dot{\gamma}| = \sqrt{\frac{1}{2} \mathbf{I}^2} = \left[ \frac{1}{2} \mathbf{I} : \mathbf{I} \right]^{1/2},
\]
where \(\mathbf{I}\) is the second invariant of \(\tau\). It follows from the
above that the criterion to track down yielded/unyielded
regions is for the material to flow (yield) only when
the magnitude of the extra stress tensor \(|\dot{\gamma}|\) exceeds the yield
stress \(\tau_0\), that is, yielded: \(|\dot{\gamma}| > \tau_0\); unyielded: \(|\dot{\gamma}| \leq \tau_0\).

Beyond a threshold stress corresponding to a thresh-
old strain rate, viscosity is described by a square root rela-
tionship. The model is given by
\[
\eta = \left( \sqrt{\mu + \sqrt{\tau_0 / \dot{\gamma}}} \right)^2.
\]

Here, \(\eta\) is the viscosity, \(\tau_0\) is the yield stress, and \(\mu\) is
the viscosity coefficient of the Casson fluid. The constants
for the Casson model used in the present work were
taken from Reference \(38\) and are given in Table 1.

The curves of viscosity as a function of shear rate pre-
dicted by the constitutive equations of the models just
described are shown graphically in Figure 2. The figure
shows that the Casson model gives the highest viscosity
in the lower shear rate range, while the power law and
the Carreau model give the lowest viscosity. The Carreau
and Casson fluids coincide with the power law model in
the intermediate shear rate range \((\dot{\gamma} = 5 - 100)\), but in the
large shear rate range, the power law model does not
coincide with the Carreau and Casson models and the
viscosity decreases continuously. To achieve reasonable
agreement between the predictions of the three models,
the chosen characteristic deformation rate must fall

**TABLE 1** The values of the parameters of various general non-
Newtonian fluid models.

<table>
<thead>
<tr>
<th>GNF model</th>
<th>Parameter</th>
<th>Value</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>—</td>
<td>(\rho)</td>
<td>1050 kg.m(^{-3})</td>
<td>—</td>
</tr>
<tr>
<td>—</td>
<td>(D)</td>
<td>0.01 m</td>
<td>—</td>
</tr>
<tr>
<td>Power law model</td>
<td>(k)</td>
<td>0.017 Pa.s(^{a})</td>
<td>36</td>
</tr>
<tr>
<td></td>
<td>(n)</td>
<td>0.708</td>
<td>36</td>
</tr>
<tr>
<td>Bird–Carreau</td>
<td>(\lambda)</td>
<td>3.313s</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>(a)</td>
<td>2</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>(n)</td>
<td>0.3568</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>(\eta_0)</td>
<td>0.00345 Pa.s</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>(\eta_\infty)</td>
<td>0.056 Pa.s</td>
<td>37</td>
</tr>
<tr>
<td>Casson model</td>
<td>(\tau_0)</td>
<td>0.0108 Pa</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>(\mu)</td>
<td>0.00276 Pa.s</td>
<td>38</td>
</tr>
</tbody>
</table>
within the range of deformation rates actually observed in the flow. For predominantly shear flows of non-
Newtonian fluids with shear rate dependent viscosities, a
choice in the middle range of shear rates, as seen in the
viscosity function of Figure 2, would be appropriate.
Assuming the same Reynolds number for all models,
namely Re = 100, we obtain the characteristic values for
velocity, shear rate, and viscosity given in Table 2 where
we see that the viscosity hardly changes and \( \dot{\gamma}_c \) is in the
intermediate region seen in Figure 2.

2.2 | Boundary conditions

The schematic of the two-dimensional cross-slot with a
pair of inlet and outlet channels of equal length \( l \) and
width \( D \) is shown in Figure 1. At \( x = 0 \), the two inlet
streams meet in the center of the geometry and are
diverted perpendicularly into the two identical outlet
arms. A uniform velocity \( u_0 \) is applied on the inlets in
such a way that the inflow Reynolds number based on the
velocity \( u_0 \), the characteristic viscosity \( \eta_c = \eta(\dot{\gamma}_c) \), and
the width \( D \) is \( Re = 100 \). Neumann boundary conditions
are considered for all variables on the outlets. In hemody-
namics, the Reynolds number for blood flow in arteries
and veins is typically in the range of 1000 or lower.\(^{40}\)
\( Re = 100 \) can be considered a representative of the blood
flow in smaller vessels and capillaries. The length of the
inlet and outlet arms is fixed at \( 15D \) (with \( D = 0.01 \) m),
which is sufficient for the flow to be fully developed
upstream of the cross-slot. At the walls, no-slip bound-
ary conditions are applied and additionally the walls
are considered electrically insulated, that is, the electric
current cannot penetrate through them. These
boundary conditions, in general form, are given by:

\[
\begin{align*}
\text{Inlets} : & \quad u = u_0, \quad \frac{\partial p}{\partial n} = 0, \quad \phi = 0 \\
\text{Outlets} : & \quad \frac{\partial u}{\partial n} = 0, \quad p = 0, \quad \phi = 0 \\
\text{Walls} : & \quad u = 0, \quad \frac{\partial p}{\partial n} = 0, \quad \frac{\partial \phi}{\partial n} = 0.
\end{align*}
\]

2.3 | Discretization schemes and solutions algorithm

In the present study, the proposed numerical method is
based on the previous work of Dousset and Pothérat.\(^{41}\)
The numerical simulation consists in solving the governing
Equations (1)–(5) using the OPENFOAM software
and is based on the finite volume approach. The steady-
state flow is simulated in a time marching way, by
advancing the velocity and pressure fields as a function
of time. Below are some details about the selected discret-
ization schemes used in this study. The first-order Euler
scheme is used to discretize the time derivative terms.
The time step is calculated automatically during the sim-
ulations using the Courant number criteria. By fixing
\( \text{Co}_{\text{max}} = 1 \), the time step is computed from the minimum
value of \( (\Delta x / u_m) \) taken over all fluid cells where \( \Delta x \) is
the grid spacing and \( u_m \) is the magnitude of flow velocity.
The divergence term in the momentum equation is dis-
cretized with a Gauss linear scheme, which can be first
or second order and is bounded for stability. The Lapla-
cian term uses a Gauss linear corrected scheme, which is
second order and the gradient term uses a least squares
scheme that is also second order. Interpolation from cell
centers to face centers is based on the linear scheme and
the surface normal gradient term use the corrected
scheme, where an explicit non-orthogonal correction is
applied.

The pressure–velocity coupling is performed by using
the pressure implicit split operator (PISO) algorithm to
obtain the pressure and velocity. Poisson’s Equation (5) is
solved to calculate \( \phi \), and finally, the current density in
each cell is reconstructed to obtain the Lorentz force
\( f = j \times B \) and the source term in the momentum
equations. The previous procedure is repeated iteratively to

![FIGURE 2 Variation of viscosity with shear rate for all non-
Newtonian models.](https://example.com/figure2.png)

<table>
<thead>
<tr>
<th>Models</th>
<th>Power law</th>
<th>Bird–Carreau</th>
<th>Casson</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta_c (\text{Pa.s}) )</td>
<td>0.00604</td>
<td>0.00596</td>
<td>0.00514</td>
</tr>
<tr>
<td>( \dot{\gamma}_c (1/\text{s}) )</td>
<td>34.5337</td>
<td>34.0770</td>
<td>29.3827</td>
</tr>
</tbody>
</table>

**TABLE 2** Characteristic velocity, shear-rate, and viscosity for the non-Newtonian models (Re = 100).
ensure accuracy and consistency of the velocity and electric current fields; a normalized convergence criterion of $10^{-5}$ is applied for the pressure, velocity, and electric potential.

3 | VALIDATION AND MESH ANALYSIS

The accuracy of the present computational model is examined against the results of Sarkar and Ganguly\textsuperscript{42} (first validation) and Sarpkaya\textsuperscript{20} (second validation). For the first validation case, the fully developed flow of a non-Newtonian fluid in a two-dimensional parallel-plate microchannel is considered, and the simulation results are compared with those of Sarkar and Ganguly.\textsuperscript{42} In this study, they consider the axially pressure-driven flow of a non-Newtonian fluid as shown in the inset of Figure 3A. For this case, a uniform pressure is given at the inlet. The Reynolds number for this flow is low (Re $< 1$), typical for narrow fluidic confinements, and it is defined in terms of the imposed pressure gradient $\frac{dp}{dx}$ using a reference velocity $u_0 = -\frac{dp}{dx} \frac{D^2}{\eta}$. The viscosity of the working fluid is represented by the power law model with two values of the power law index $n = 0.5$ and $n = 1.5$. A uniform magnetic field $B_0$ is imposed perpendicular to the channel walls resulting in $Ha = 6$. The length of the microchannel is assumed to be $15D$, and calculations are performed with a grid size of $101 \times 51$. The comparison of the analytical solution of Reference 42 and our numerical results of normalized velocity profiles for the shear thinning and shear thickening conditions is shown in Figure 3A. In this figure, the velocity profile is plotted along a vertical line sufficiently downstream of the inlet, for example, $x = 7.5D$. It is observed that there is a very good agreement between the simulation results and the benchmark analytical results. This gives us confidence in the validity of our numerical model and in its accuracy for other ranges of parameters that are considered in the next section.

For the second case, the two-dimensional non-Newtonian MHD channel flow is simulated with a power law model, and the results are compared with available analytical data.\textsuperscript{20} The details of the physical model and its boundary conditions are the same as given in the inset of Figure 3B. No slip walls with distance $y = \pm h$ are considered as subjected to the boundary conditions with a centerline symmetry. The length of the channel is assumed to be equal to $30h$. The velocity $u$ is in the x-direction and a uniform magnetic field $B_0$ is imposed perpendicular to the channel walls. The properties of the working fluid is represented with power law index $n = 0.5$. Calculations are performed with a grid size of $101 \times 51$ for two values of the magnetic field. Figure 3B shows the comparison of the axial velocity component ($u/\nu_M$, where $\nu_M$ is here the centerline velocity) with Sarpkaya\textsuperscript{20} for magnetic parameters equal to $Ha = 0$ and $Ha = 6$. The velocity profile is at a station sufficiently downstream of the inlet, namely $x = 15h$, in such a way for the flow to be fully developed. It is clear that the reported results show an excellent agreement with the analytical study.

We turn now to the cross-slot flow and the issue of convergence with mesh refinement. To ensure mesh

![Figure 3A](https://example.com/figure3a.png) ![Figure 3B](https://example.com/figure3b.png)

FIGURE 3 Comparison of the present MHD non-Newtonian flow numerical model with: (A) the results of Sarkar and Ganguly\textsuperscript{42} for power law fluid with different power law index; (B) the results of Sarpkaya\textsuperscript{20} for various magnetic field strength.
independence, solutions were obtained on three consecutive refined grids, denoted M1, M2, and M3. Figure 4 illustrates the cross-slot geometry with a structured mesh applied to all parts of the computational domain. Each channel has an increased mesh density closer to the cross-slot junction and a uniform cell size is chosen in the central square. The total number of grids \( (N) \) inside the computational domain and the nominal non-dimensional grid spacing \( h \) are given in Table 3.

To evaluate the accuracy of the mesh refinement results, averages viscosity \( \langle \eta / \eta_c \rangle_{avg} \) along the vertical line at \( x = -D \) for three different mesh sizes are given in Table 3 for the power law, Bird-Carreau, and Casson non-Newtonian fluid models. A grid convergence index (GCI) was calculated according to Reference 43 for the results shown in Table 3. GCI uses the Richardson extrapolation for estimation of the numerically converged solution (the solution obtained when the grid spacing tends to zero, \( h \to 0 \)). This GCI provides an estimate of the amount of error between the finest grid and the numerically converged solution. In our case, GCIs are calculated for the very fine mesh (M3) and the fine mesh (M2). In both cases, the GCI value compares the error between the results of a specific mesh with the numerically converged solution. From the results of Table 3, with the GCI based on M3, we see that the maximum value of this index is less than 1%, which shows very negligible variation of the results with the grid. On the basis that mesh 3 (very fine mesh) is favorable in terms of GCI, M3 is selected as the grid that provides grid-independent results for all remaining computations.

4 | RESULTS AND DISCUSSION

Depending on the scientific and technological applications, fluids with different properties are needed in different real-life situations. When viscosity is the main property to characterize a given flow behavior, then inelastic rheological models may be sufficient for certain applications. Here three GNF models defined in Section 2 were selected to represent the variation of viscosity with shear rate, namely the power law, Bird–Carreau, and Casson models. Numerical simulations were performed to analyze the effect of magnetic field, measured by Hartmann number (Ha), on a two-dimensional cross-slot flow with non-Newtonian fluid. The schematic of the cross-slot geometry is given in Figure 1. Detailed kinematics of the flow are discussed for a moderate Reynolds number (\( Re \approx 100 \)) based on the numerical results for the velocity field and shear viscosity patterns.

Figure 5 shows the effect of the magnetic field on the non-dimensional velocity magnitude \( \mu / \mu_0 \) and normalized shear viscosity \( \eta / \eta_c \) for the Bird–Carreau model. It

![Figure 4](https://example.com/figure4.png)

**Figure 4** Schematic of the cross-slot computational grid.

**Table 3** The results of grid study for average velocity and viscosity at section \( x = -D \) for different mesh size with \( Ha = 0 \) and \( Re = 100 \).

<table>
<thead>
<tr>
<th>GNF model</th>
<th>Mesh</th>
<th>( N )</th>
<th>( h )</th>
<th>( \langle \eta / \eta_c \rangle_{avg} )</th>
<th>GCI(( \eta / \eta_c ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power law</td>
<td>M1</td>
<td>15,525</td>
<td>0.04</td>
<td>1.5042</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>M2</td>
<td>63,597</td>
<td>0.02</td>
<td>1.4961</td>
<td>0.024</td>
</tr>
<tr>
<td></td>
<td>M3</td>
<td>257,397</td>
<td>0.01</td>
<td>1.4958</td>
<td>&lt;10^{-3}</td>
</tr>
<tr>
<td>Bird–Carreau</td>
<td>M1</td>
<td>15,525</td>
<td>0.04</td>
<td>1.8852</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>M2</td>
<td>63,597</td>
<td>0.02</td>
<td>1.8632</td>
<td>0.314</td>
</tr>
<tr>
<td></td>
<td>M3</td>
<td>257,397</td>
<td>0.01</td>
<td>1.8594</td>
<td>0.055</td>
</tr>
<tr>
<td>Casson</td>
<td>M1</td>
<td>15,525</td>
<td>0.04</td>
<td>2.0246</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>M2</td>
<td>63,597</td>
<td>0.02</td>
<td>1.9856</td>
<td>0.348</td>
</tr>
<tr>
<td></td>
<td>M3</td>
<td>257,397</td>
<td>0.01</td>
<td>1.9807</td>
<td>0.043</td>
</tr>
</tbody>
</table>
FIGURE 5  Superimposed streamline patterns and contour plots of velocity magnitude for (A) $Ha = 0$, (B) $Ha = 10$, and (C) $Ha = 20$; and contour plots of viscosity for (D) $Ha = 0$, (E) $Ha = 10$, and (F) $Ha = 20$ (Bird–Carreau model).
illustrates how the inlet flow streams after the double bifurcation imposed by the geometry at the central region of the cross-slot are split into two equal parts and how then they flow identically along the two outlet ports. Well-defined bi-symmetric flow patterns are observed for both the velocity and viscosity contours of the non-Newtonian fluid, with symmetry occurring about both the $y=0$ plane and the $x=0$ plane. From Figure 5A–C, it can be seen that the velocity field tends to zero as the fluid approaches the central region due to the presence of the stagnation point at the center of the cross-slot, and that such well-defined central stagnant circular region tends to increase in size as $\text{Ha}$ increases. In Figure 5A, the velocity in the middle of the inlet channels is higher than in the regions closer to the adjacent walls of the cross-slot because the influence of the Lorentz force in the flow is zero due to the absence of magnetic field. Moreover, due to the effect of inertia, which is enhanced by local low viscosities for this GNF fluid, a tendency for flow separation is observed near the four corners in the outlet arms: the streamlines first converge and then diverge as the fluid turns the re-entering corners (we recall $\text{Re} \approx 100$). For higher Reynolds number, there is actual flow separation with the formation of lip vortices seen as small recirculating eddies near the corners of the outlet arms. A small zone of low velocity is observed in Figure 5A for $\text{Ha} = 0$ next to each corner. When the magnetic field is turned on, this effect is reduced. Increasing the magnetic field ($\text{Ha} = 10$) causes a retardation effect on the flow in the inlet channels which significantly suppresses the flow in the core regions. Since the magnetic field is imposed perpendicular to the x-axis, it results in a force per unit volume of $-\mathbf{B}_0 \times \mathbf{E}$ (with $\mathbf{E}$ being the unit vector in the x-direction and $u_x$ being the x-component of the velocity). As a result, the flow intensity in the middle of the inlet arms is reduced, which has a strong effect on the flow in the central square region and the flow along the initial part of the outlet arms. At the same time, it is observed that the velocity of the fluid along the edges of the outlet ports starts to increase when $\text{Ha}$ is increased, which is due to the high velocity regions induced by the magnetic field near the walls of the inlet ports. When the Hartmann number is further increased ($\text{Ha} = 20$), the fluid reaches maximum velocities in the regions adjacent to the walls, which can be clearly seen in the velocity contours of the outlet sections. For higher Reynolds numbers, this could suppress the formation of vorticities near the corners of the outlet arms.

The viscosity contours in Figure 5D–F reflect the velocity distribution just described. In the absence of a magnetic field (Figure 5D), viscosity is large in the central region of the channels, where the velocity profile is flat and is reduced by shear thinning along the layers adjacent to the walls. Some lack of symmetry between the incoming and outgoing flows is evident due to inertia, as the former slows down when the stagnation region is approached and its effect becomes noticeable and the latter accelerates along the outlet branches. In the presence of an external magnetic field, the high viscosity zone in the inlet arms is enhanced and tends to extend across these channels leading to high shear rates flow zones concentrated near the walls. For the $\text{Ha} = 10$ case, the viscosity patterns become thicker in the middle part of the inflow channel while increasing the magnetic field. This means that the inertial force of the fluid is considerably reduced as compared to the viscous force. Therefore, the velocity profile corresponding to the two flow layers leaving the central region of the cross-slot and extending to the center of the outlet channels become flat (low $\dot{\gamma}$) and hence the corresponding viscosity concentration regions becomes enhanced. When the magnetic field is increased to $\text{Ha} = 20$, the fast-moving flow near the walls of the outlet channels is stronger.

Streamwise velocity profiles for different magnitudes of the applied magnetic field, as measured by Hartmann number ($\text{Ha}$), are shown in Figure 6 for two physically opposite scenarios: the inlet flow immediately after the fully developed region and approaching the central square region, at $x = -D$, and the flow scenario shortly after entry into the outlet channel, at $y = D$. For $\text{Ha} = 0$, it can be seen from Figure 6A–D that the velocity distribution in the central region is somewhat more uniform compared with the Newtonian parabolic profile having a maximum $u_{\text{max}} = 1.5$ at centerline, in both the inlet and the developed velocity profiles that can be derived for the GNF models considered here. Similar profiles are obtained for these non-Newtonian models except for larger velocities in the central part of the channel and slight decreased near the walls for the power law model. For $\text{Ha} > 0$, the first thing to notice is that the body force in the momentum transport equation [Equation (2)] has component acting in opposite direction of the flow due to the imposed electromagnetic field and that plays a crucial role in the flow field. Therefore, in such situation, as the Hartmann number increases, the velocity plateau in the middle of the inlet channels is gradually enhanced due to the adverse effect of the electromagnetic body force (see Figure 6B for $\text{Ha} = 10$) and larger velocity gradients and consequently smaller viscosities develop near the walls of the inlet channels as they approach the re-entrant corners (Figure 5E). For $\text{Ha} = 10$, the velocity profile has essentially a rectangular shape with $u = 1.17u_0$ in the center and a narrow Hartmann layer near the walls. In MHD, the Hartmann layer refers to a kind of boundary.
FIGURE 6  Velocity profiles at $x = -D$ in the inlet channel (left column), and at $y = +D$ in the entrance region of the outlet channel (right column) for different values of magnetic field with various non-Newtonian models.
layer formed in a conducting fluid flowing in the presence of a magnetic field. This layer exhibits suppressed fluid motion perpendicular to the applied magnetic field due to the Lorentz force that flattens the velocity profile. Such tendency is accentuated for the larger value of $Ha = 20$, the velocity level in the plateau becomes $u \approx 1.09u_0$ and therefore the uniform velocity tends to get closer to the walls to conserve mass for the flow entering the cross-slot channels (Figure 6C). In fact, the thickness of the Hartmann layer depends on the magnetic field strength according to the relation $\delta_H \propto D/ Ha$ where $D$ is the depth of the channel. This shows that doubling the Ha number leads to a reduction of the thickness of the Hartmann layer by a factor of 2 (as in Figure 6B,C). The outlet velocity profiles (Figure 6D–F) exhibit a consistent trend, following the increase of the Hartmann number (Ha), in comparison with the inlet velocity profiles. This behavior is attributed to the influence of the magnetic field, where the central region experiences a reduction in velocity, leading to flattened velocity profiles. Notably, the outlet ports consistently display higher maximum velocity values compared to the inlet ports across all Hartmann numbers. The inlet velocity profile exhibits a parabolic behavior, while the outlet ports showcase bell-shaped velocity patterns for all non-Newtonian models. This difference is due to the absence of Hartmann layers in the outlet channels attributed to the orientation of the magnetic field. The magnetic field is more dominant in the inlet arms, with the Lorentz force acting perpendicularly to the inlets and parallel to the outlet ports. As a result, the velocity profiles are more significantly impacted on the inlet arms compared to the outlets. While the formation and evolution of Hartmann layers can be seen in the inlet channels in Figure 6B,C, the velocity distribution in the outlet channels (Figure 6E,F) is a consequence of Hartmann layers near the inlet channels walls at the corners, coupled with shear thinning. In conclusion, when the Hartmann number is further increased, the reduction in flow velocity is more pronounced in the center of the channels compared to the flow near the walls. In relation to this, we recall that the magnetic body force varies with $B^2$ and so the effects on the flow field are not linear with the values of Ha. There are no significant differences between the results for the different non-Newtonian models with respect to the streamwise velocity distribution in the inlet and outlet ports of the cross-slot.

In Figure 7, the streamwise velocity profiles of the Casson model at $y = -D$ are plotted to enable distinguishing the effects of a Hartmann layer in the velocity profiles from the typical plug-flow-like profile observed in yield-stress fluids. In Figure 7, the formation of the Hartmann layer when external magnetic field is active can be observed. The thickness of the Hartmann layer is well scaled with $\delta_H \propto D/ Ha$. For the case with $Ha = 0$, such layer is not formed and only a plug flow region in the center of the channel with thickness $y_0$ is observed. The thickness $y_0$ can be estimated assuming a fully developed flow in channel $(u = u(y), p = p(x))$ for $Ha = 0$ following the same procedure presented in Sarpkaya.\textsuperscript{20} In this case, analytical solution of the thickness of the plug flow will lead to $y_0 = r_0/\left(-\frac{dp}{dx}\right)$. It should be mentioned that a key distinguishing factor lies in the nature of velocity gradients. In the Hartmann layer, velocity gradients are reduced near the walls, leading to a suppressed flow. In yield-stress fluids, the rapid decrease in velocity is often attributed to the yield stress being exceeded near the walls. Figure 7 illustrates the plug-flow velocity profile without a magnetic field, and also demonstrates the formation and evolution of Hartmann layers when a magnetic field is imposed. The presence of the magnetic field leads to rapid and sharp velocity gradients near the walls, coupled with a significant flattening of the velocity profiles at the center of the channel. It is important to refer here that the thickness of a Hartmann layer, especially for large Ha, is much smaller than the thickness of the wall layer of a non-Newtonian shear-thinning fluid, even one with a yield stress at the Casson model: suffices to compare the wall velocity gradient in Figure 7 for the case without magnetic field and the case with $Ha = 20$. It is also crucial to emphasize that the observed sharp decrease in velocity near the walls and the flatter region in the center, while resembling turbulent-like profiles with logarithmic or power law behavior, originates from a fundamentally different mechanism than turbulent
flows. Turbulent flows involve the formation of eddy structures and typically exhibit high shear rates, especially near the walls dominated by viscous forces. The fluid near the walls experiences significant shear stress due to rapid changes in velocity, primarily induced by turbulent vortices.

Figure 8 shows the distribution of the normalized viscosity resulting from the different non-Newtonian models in the inlet and outlet ports at $x = -D$ and $y = D$, respectively, for the same physically opposite scenarios we presented in the previous paragraph, with increasing values of the magnetic field. In Figure 8A for $H_a = 0$, the power law and Bird–Carreau models have lower viscosity values in the middle of the channel compared with Casson model predictions as a result of the strain rate that the flow experiences on approaching the stagnation region ($\dot{\gamma} = 10^{-1} - 3 \times 10^{-1} \text{s}^{-1}$, cf. Figure 2). In Figure 8B, the presence of a magnetic field with $H_a = 10$ brings in the effect of the flow-opposing Lorentz force which indirectly flattens the viscosity profiles, thus enhancing the average viscosity across the inlet channels, more strongly for the Casson model, followed by the Bird–Carreau and power law models. Thereafter, at $H_a = 20$ and the inlet port (see Figure 8C), a viscosity profiles with two peaks is observed. In addition, the viscosity for all non-Newtonian fluids is increased in the middle of the channel. This is due to the magnetic force, which causes the velocity profiles to become more uniform. The viscosity profiles with two peaks are a consequence of increased pressure generated by the enlarged size of the central stagnation point for $H_a = 20$. The fluid deceleration in the Hartmann's layers close to the walls of the two inlet arms and flattened velocity profiles at the central region of the channel contribute to a two-peak distribution of the strain rate. Specifically, the two minimum strain rate values are obtained between the centerline and walls, with local maxima at the center of the channel. This results in a two-peak distribution of viscosities. In this case, the viscosity of the Casson model reaches the unyielded region in two regions near the central part of the channel, where the maximum viscosity is about 10.9 times its characteristics value and $\dot{\gamma} \approx 0$. On the other hand, in the central part of the channel $|y| \leq 0.1$ (Figure 8C), the extensional component of $\dot{\gamma}$ generated by the stagnation region makes $\dot{\gamma}$ to rise, the viscosity diminishes, and the fluid yields and flows. In the outlet port and in the absence of a magnetic field, the viscosity distribution for all non-Newtonian models has a maximum in the middle of the channel where the velocity gradients are smaller, with the power law having smaller viscosities, as shown in Figure 8D. When the magnetic field is turned on, giving $H_a = 10$ (Figure 8E) and $H_a = 20$ (Figure 8F), the distribution of viscosity at the entrance to the outlet arms is determined by the fluid acceleration in the Hartmann layers near the walls of the two inlet arms. This results in thicker layers of fast fluid near the walls in the outlet channels. Using the Casson model, the case with $H_a = 10$ leads to an unyielded region where the viscosity distribution is flat at its highest value $\eta / \eta_0 = 10.9$ in the middle of the channel (Figure 8E). When $H_a$ is increased to $H_a = 20$, the flow acceleration near the walls is increased and the size of the central stagnation region is enlarged enough so that, together with the usual local velocity maximum at the channels centerline, leads to a two peaks distribution of viscosity. The larger velocities that occur near the walls for the Casson model prevent the formation of the unyielded region, except in the center of the channel where such a region is expected further downstream similarly to that seen in the inlet channels.

In Figure 9, we plot the variation of the normalized velocity magnitude along the centerline obtained using the Bird–Carreau model for various Hartmann numbers, near the central stagnation region. It shows that the approaching flow to the central stagnation point (at $x = y = 0$) differs from the outflow, starting from the stagnation point, as a consequence of the inertia considered here ($Re = 100$). We recall that the velocity is made dimensionless by scaling with the corresponding average inlet velocities. As the magnetic field is increased, the velocity along the centerline tends to be reduced due to the opposing electric/magnetic force. In spite of this, the rate of velocity decay on approaching the stagnation point is similar for the three cases of $H_a = 0, 10,$ and $20$. It is also apparent that the flow in the central region affects the upstream conditions up to approximately $x \approx 2D$. Figure 9B shows that the pattern of outflow velocity along the centerline is similar for the $H_a = 0$ and $H_a \neq 0$ cases, specifically in the acceleration region $|y/D| \leq 1$. Namely, without magnetic field ($H_a = 0$), the outflow acceleration occurs along the centerline of the outlet channels, with a visible velocity overshoot resulting from the “vena contraction” effect already commented in relation to Figure 5A. However, in the presence of a magnetic field, the flow is accelerated also near the walls of the outlet channels, as the strong incoming fluid layers adjacent to the wall generated by electric/magnetic effects in the inlet channels turn around the four cross-slot corners. For the case of $H_a = 20$, it is only quite further downstream along the outlet channels that the maximum velocity moves to the center-line.

The variation of viscosity along the centerline of the inlet (Figure 10A) and outlet (Figure 10B) channels for the Carreau model confirms the previous observations about the velocity patterns. In regions of fully developed flow, $\eta$ should be large in the center of the channels, because the shear rate there is zero, as seen in
Figure 8: Profiles of shear-viscosity at $\frac{y}{D} = \pm 1$ in the inlet channel (left column) and at $\frac{y}{D} = \pm 1$ in the entrance region of the outlet channel (right column) for different values of magnetic field and non-Newtonian models.
Figure 10A for $|x/D| \geq 2$. Additionally, Figure 10A shows that the flow penetrates much further into the central region of the cross-slot without being distorted when the magnetic field is present (from $|x/D| = 2$ to 1). With an applied magnetic field, the maximum velocity in the flat region of the velocity profiles is reduced, thus resulting in attenuation of inertia and consequently the flow reaches the central region slightly farther upstream (at $x \approx 1.5D$ for the $H_a = 10$ and $x \approx 1.0D$ for the $H_a = 20$). These effects are seen more clearly with the viscosity variation of Figure 10A than with the velocity variation of Figure 9A. Along the outlet channels, Figure 10B reveals by means of the viscosity variation in the centerline, the presence of internal points where the shear rate is zero ($\dot{\gamma} = 0$) for $H_a = 0$ and 10 at $y \approx \pm D$. At these points, the shear viscosity attains its maximum value, which is $\eta_s/\eta_c = 9.40$ with $\eta_c$ given in Table 2. The fact that locally we have $\dot{\gamma} = 0$ (implying $\partial v/\partial y = 0$) is explained by the local maximum (overshoots) of velocity variation seen in Figure 9B (only for $H_a = 0$ and 10). At the highest $H_a = 20$, Figure 9B does not show an overshoot in the values of velocity variation and so $\dot{\gamma}$ is never locally zero along the centerline of the outlet channels. This is a direct consequence of the change in flow patterns discussed earlier: fast-moving fluid layers are more deflected toward the walls in the outlet channels as $H_a$ raises.

Another feature of interest is the yield stress behavior of the Casson model and how it is affected by the magnetic effect. Figure 11 shows the flow behavior and the corresponding evolution of the yielded/unyielded regions for the Casson fluid at different magnetic field values.

For $H_a = 0$ (Figure 11A), plug regions are clearly visible in the central part of the upstream and downstream channels forming the cross-slot, where the flow is fully developed and the velocity profile is flat in the center ($y = 0$). The unyielded (solid-like) regions are represented in the contour plots by the solid blue color (noting that $\tau_c/\tau_w = 0.428$), while the other regions correspond to yielding regions (i.e., deforming fluid). From Figure 11B, it can be seen that as the Hartmann number increases, the unyielded regions in the inlet arms increase, corresponding to a solid-like structure supported by a very thin layer of fluid near the walls of the cross-slot (the Hartmann layers). Plug flow occurs in the upstream and downstream parts of the channels, which are truly unyielded regions and move with a plug velocity profile (i.e., without deformation) where the velocity gradients are very small due to $|\tau| < \tau_c$. When $H_a$ increases again (Figure 11C for $H_a = 20$), the unyielded regions in the incoming channels become larger occupying most of the inlet region and form a solid-like plug flow in most of the channel cross-section.

In Table 4, non-dimensional pressure drop values ($\Delta p' = \Delta p/(0.5\mu_c^2 \rho_c^2 L)$) are presented for various non-Newtonian model tested here under the influence of a magnetic field. The pressure drop is non-dimensionalized using the inlet velocities ($u_0$) and the equivalent length of the channel ($L$), defined as the distance from one inlet to outlet calculated at the center of the channel. Normalized in this way, $\Delta p'$ may be interpreted as half of a skin friction coefficient, $C_f = \frac{4\tau_w}{0.5\rho_0 u_0^2} = 2\Delta p'$, where $\tau_w$ is an average wall shear stress. The results indicate a non-linear
FIGURE 10  Variation of viscosity along the centerline of (A) the inlet channel and (B) outlet channel for different values of magnetic field with Bird–Carreau model. (A) Horizontal centerline ($y = 0$). (B) Vertical centerline ($x = 0$).

FIGURE 11  Contour plots of the magnitude of the stress tensor for the Casson model with different values of magnetic field.

(A) $Ha = 0$  
(B) $Ha = 10$  
(C) $Ha = 20$
increase in pressure drop with the Hartman number (Ha), where higher Hartman numbers correspond to increased pressure drop. Notably, the pressure drop values for all the GNF models closely align with each other, a result that may be appreciated by the correct scaling using characteristic viscosities given in Table 2. Specifically, for all of the GNF models, the pressure drop increases by approximately 4.80 times when the Hartman number is raised to 10. However, doubling the Hartman number (Ha = 20) results in a 3.26-fold increase in pressure drop. The values of the pressure drop, particularly when expressed as ratio to the corresponding pressure drop of a Newtonian fluid \( \Delta p' = \frac{\Delta p}{\Delta p_N} \), are indicative of the pressure losses or parasitic pumping needed to drive the non-Newtonian fluid through the cross-slot. Thus, these findings suggest that while an external magnetic field can alter the flow characteristics of non-Newtonian fluids, the magnetic field running transverse to the inlet channels, is the primary effect of the magnetic field applied perpendicular to the inlet channels, due to the Lorenz force, to exert an opposing force on the two incoming flows. This results in flatter velocity profiles, with a lower velocity in the middle section and a flow acceleration layer near the walls of the inlet channels. The initial consequence of this effect is a reduction in flow momentum as the central stagnation point is approached. The same trend of flattening the velocity profiles had been previously reported for Newtonian and non-Newtonian flow in a straight channel under the effect of magnetic field. Consequently, the stagnation point becomes noticeable further upstream when Ha is raised, and a greater distortion of the incoming and outgoing flow is expected. This change in the size of the central stagnation region in a cross-slot under the influence of magnetic field had not been reported in the literature. A second result, which can actually be considered a secondary effect of the magnetic field running transverse to the inlet channels, is the development of a kind of wall jets along the outlet channels of the cross-slot. There is little difference between the three non-Newtonian models that were fitted to give the same viscosity at a given characteristic shear rate. However, as a function of Hartmann number, the viscoplastic Casson model gives rise to un-yielded regions in the middle of the inlet channels. These features are not observed in the other models, which are clearly shear-thinning. The results for the Casson model also indicate that the flow kinematics of the yield stress is significantly affected by the magnetic field. This observation has a good agreement with previous study of MHD flow on Bingham fluid in a channel flow.\(^{46-48}\) The conclusions of the present study are presented below.

The primary effect of the magnetic field applied perpendicular to the inlet channels, due to the Lorenz force, is to exert an opposing force on the two incoming flows. This results in flatter velocity profiles, with a lower velocity in the middle section and a flow acceleration layer near the walls of the inlet channels. The initial consequence of this effect is a reduction in flow momentum as the central stagnation point is approached. The same trend of flattening the velocity profiles had been previously reported for Newtonian and non-Newtonian flow in a straight channel under the effect of magnetic field. Consequently, the stagnation point becomes noticeable further upstream when \( Ha \) is raised, and a greater distortion of the incoming and outgoing flow is expected. This change in the size of the central stagnation region in a cross-slot under the influence of magnetic field had not been reported in the literature. A second result, which can actually be considered a secondary effect of the magnetic field running transverse to the inlet channels, is the development of a kind of wall jets along the outlet channels as the Hartmann layers formed along the inlet arms pass around the cross-slot corners during the transition from the inlet to the outlet channels. Immediately after the cross-slot, in the outlet arms, the velocity profiles lead to a 2-peak viscosity profile (with a minimum in the center of the outlet channels and two maxima near the walls where the peak velocities occur, with \( j \) approaching zero, cf. Figure 8F). It can be concluded that the magnetic field, which is proportional to the Ha number, causes mixing in the outlet channels of the cross-slot. There is little difference between the three non-Newtonian models that were fitted to give the same viscosity at a given characteristic shear rate. However, as a function of Hartmann number, the viscoplastic Casson model gives rise to un-yielded regions in the middle of the inlet channels. These features are not observed in the other models, which are clearly shear-thinning. The results for the Casson model also indicate that the flow kinematics of the yield stress is significantly affected by the magnetic field. This observation has a good agreement with previous study of MHD flow on Bingham fluid in a channel flow.\(^{35}\) The conclusions of the present study are presented below.

The primary effect of the magnetic field applied perpendicular to the inlet channels, due to the Lorenz force, is to exert an opposing force on the two incoming flows. This results in flatter velocity profiles, with a lower velocity in the middle section and a flow acceleration layer near the walls of the inlet channels. The initial consequence of this effect is a reduction in flow momentum as the central stagnation point is approached. The same trend of flattening the velocity profiles had been previously reported for Newtonian and non-Newtonian flow in a straight channel under the effect of magnetic field. Consequently, the stagnation point becomes noticeable further upstream when \( Ha \) is raised, and a greater distortion of the incoming and outgoing flow is expected. This change in the size of the central stagnation region in a cross-slot under the influence of magnetic field had not been reported in the literature. A second result, which can actually be considered a secondary effect of the magnetic field running transverse to the inlet channels, is the development of a kind of wall jets along the outlet channels as the Hartmann layers formed along the inlet arms pass around the cross-slot corners during the transition from the inlet to the outlet channels. Immediately after the cross-slot, in the outlet arms, the velocity profiles lead to a 2-peak viscosity profile (with a minimum in the center of the outlet channels and two maxima near the walls where the peak velocities occur, with \( j \) approaching zero, cf. Figure 8F). It can be concluded that the magnetic field, which is proportional to the Ha number, causes mixing in the outlet channels of the cross-slot. There is little difference between the three non-Newtonian models that were fitted to give the same viscosity at a given characteristic shear rate. However, as a function of Hartmann number, the viscoplastic Casson model gives rise to un-yielded regions in the middle of the inlet channels. These features are not observed in the other models, which are clearly shear-thinning. The results for the Casson model also indicate that the flow kinematics of the yield stress is significantly affected by the magnetic field. This observation has a good agreement with previous study of MHD flow on Bingham fluid in a channel flow.\(^{35}\)
Furthermore, the present study shows that the pressure drop in the MHD flow is more pronounced as compared to the hydrodynamic case without magnetic field. Similar behavior was observed in previous studies in the literature. Additionally, we have also observed that the pressure drop across the cross-slot exhibits a non-linear increase with the Hartmann number (Ha). Higher Ha values correlate with elevated pressure drops, indicating increased parasitic pumping required to drive the non-Newtonian fluid through the cross-slot. The pressure drop values for the various GNF models are almost coincident, emphasizing that it is the impact of the magnetic field on the flow characteristics the main cause for the increased pressure drop. These results indicate that a magnetic field can be used to alter the flow characteristics of non-Newtonian fluids at the expense of more pumping power.

Although our results are general, the viscosity functions were chosen to replicate the rheology of blood. In hemodynamics, the Reynolds number for blood flow in arteries and veins is typically in the range of Re <1000 and depends on the local conditions and geometry of the blood vessels. In addition, since blood displays shear-thinning property, we have used here some generalized inelastic models to simulate non-Newtonian blood flows. However, at least under certain flow conditions, it is reasonable to expect blood to behave like a viscoelastic fluid. For our future work, analysis of the influence of the MHD on the non-Newtonian flow for a range of the Reynolds number between 1 and 1000 using viscoelastic non-Newtonian models will be targeted.

**NOMENCLATURE**

- \( l \) length of the inlet and outlet arms (m)
- \( Co \) Courant Number
- \( D \) width of the inlet and outlet ports (m)
- \( D \) rate of strain tensor (s\(^{-1}\))
- \( B \) magnetic field vector (A/m)
- \( B_0 \) Hartman number, \( Ha = B_0 D / \sqrt{\sigma / \eta_c} \)
- \( f \) Lorentz force, \( f = j \times B \) (N.m\(^{-1}\))
- \( j \) electrical current density vector (A/m\(^2\))
- \( k \) consistency index (Pa.s\(^n\))
- \( n \) power law index
- \( p \) pressure (Pa)
- \( Re \) Reynolds number, \( Re = \rho u_0 D / \eta_c \)
- \( Re_m \) magnetic Reynolds number, \( Re_m = \mu \eta u_0 D / \eta_c \)
- \( t \) time (s)
- \( x, y \) dimensional coordinates (m)
- \( u \) velocity vector (m/s)
- \( u_0 \) average velocity (m/s)
- \( u, v \) dimensional velocity components (m/s)

**6 | GREEK SYMBOLS**

- \( \phi \) electric potential (V)
- \( I \) identity tensor
- \( \tau \) shear stress tensor (N/m\(^2\))
- \( \tau_0 \) yield stress (N/m\(^2\))
- \( \dot{\gamma} \) shear rate (s\(^{-1}\))
- \( \lambda \) relaxation time (s)
- \( \eta \) dynamic viscosity (Pa.s)
- \( \eta_0 \) viscosity at zero shear rate (Pa.s)
- \( \eta_\infty \) viscosity at infinite shear rate (Pa.s)
- \( \rho \) density (kg/m\(^3\))
- \( \mu \) viscosity parameter in Casson model (Pa.s)
- \( \mu_m \) magnetic permeability (H.m\(^{-1}\))
- \( \sigma \) electrical conductivity (S.m\(^{-1}\))

**SUBSCRIPTS**

- \( \text{avg} \) averaged value
- \( c \) characteristic
- \( \text{min} \) minimum
- \( \text{max} \) maximum

**ACKNOWLEDGMENTS**

M. Abdollahzadeh acknowledges the support by FCT—Foundation for Science and Technology with the Project CEECIND/03347/2017/CP1472/CT0001 ([https://doi.org/10.54499/CEECIND/03347/2017/CP1472/CT0001](https://doi.org/10.54499/CEECIND/03347/2017/CP1472/CT0001)). Authors also acknowledge the support by Portuguese national funds by FCT—Foundation for Science and Technology, I.P., within the unit C-MAST—UIDB/00151/2020 ([https://doi.org/10.54499/UIDB/00151/2020](https://doi.org/10.54499/UIDB/00151/2020)) and UIDP/00151/2020 ([https://doi.org/10.54499/UIDP/00151/2020](https://doi.org/10.54499/UIDP/00151/2020)).

**DATA AVAILABILITY STATEMENT**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**ORCID**

K. Periyadurai [https://orcid.org/0000-0002-4625-0238](https://orcid.org/0000-0002-4625-0238)

J. C. Pascoa [https://orcid.org/0000-0001-7019-3766](https://orcid.org/0000-0001-7019-3766)

M. Abdollahzadehsangroudi [https://orcid.org/0000-0002-9396-3855](https://orcid.org/0000-0002-9396-3855)

P. J. Oliveira [https://orcid.org/0000-0001-9843-7558](https://orcid.org/0000-0001-9843-7558)

**REFERENCES**


