PRELIMINARY ASSESSMENT OF A NEW ALGORITHM FOR THE MHD EQUATIONS AT ALL MACH NUMBER REGIMES

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Abstract. In this paper we present a method for solving the ideal compressible MHD equations at arbitrary Mach number flows. Our method is based on the well-known PISO algorithm, which is a pressure based solver. In order to handle all the possible discontinuities that can be generated by the hyperbolic system of MHD equations the AUSM-MHD technique is applied for the flux calculation. It allows us to obtain conservative fluxes and face values for pressure that are then inserted in the solution algorithm. Regarding validation, three sets of test cases will be addressed. The first one is the 1D and 2D Alfvén waves test case, which will serve to assess the accuracy of the method in smooth flows. The second one is the rotated 1D Riemann problem, which is tackled using both 1D and 2D formulations for the MHD equations. The final test case is the well known Orszag-Tang vortex that will validate our case for complex two dimensional MHD shock interaction.
1 INTRODUCTION

Two groups of algorithms have been developed for the numerical simulation of magneto-
hydrodynamic flow (MHD). The first group comprises methods designed to deal with incom-
pressible flow [9] and the second to calculate highly compressible MHD flows [10, 13, 7, 8].
However, nowadays a great number of numerical gas dynamics codes are able to solve the com-
pressible Navier-Stokes equations at all Mach number regimes. Such algorithms are important
since in the field of aerospace engineering there is often a need to solve complex flow problems
that involve a wide range of Mach numbers. This is also true for the special case of MHD flow
in realistic Magneto-Plasma Dynamics (MPD) thruster geometries. It is known that these de-
vices can produce flows that range from the nearly incompressible to the hypersonic limit [11].
Still, there is a lack of solvers capable of solving efficiently the compressible MHD equations
for the whole range of Mach number regimes. In the present paper we discuss an arbitrary Mach
number MHD solver that is based on the PISO method. This new solver is an extension of a
previous method, developed for the Euler equations [14], to the compressible MHD equations.

In the next section we present the governing equations and in section 3 the numerical method
is described in detail together with the algorithm that couples all the unknown variables in-
volved. A special technique proposed by by Dedner et al. [4] to remove the numerical errors
arising from lack of satisfaction of Gauss law for the magnetic field, is also briefly described.

In section 4 three different sets of test cases are considered for validation. The first comprises
1D and 2D circularly polarized Alfvén waves, here we intend to validate the method for smooth
flows at low Mach number (Ma ≈ 0.2). The second set is a 1D Riemann problem, which is
tackled using the 1D and the 2D formulation of the MHD equations (obtained by rotating the
flow domain). This test case will allow us to validate the algorithm for high Mach number
flows possessing discontinuities in both velocity and magnetic fields. The final test case is the
Orszag-Tang vortex which is a standard two dimensional case for MHD schemes.

2 GOVERNING EQUATIONS

Magnetohydrodynamics is related to the interaction of a conducting moving fluid with one or
more magnetic fields. This interaction can be described by the MHD equations, which couple
the magnetic field, given by Maxwell equations, with the flow of a conducting fluid, ruled by the
Euler equations. The MHD equations for a perfectly conducting fluid, written in a conservative
form, are given by:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho U) = 0,$$

$$\frac{\partial \rho U}{\partial t} + \nabla \cdot \left[ \rho U U + \left( p + \frac{B \cdot B}{2} \right) I - BB \right] = 0,$$

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot \left[ \left( \rho e + p + \frac{B \cdot B}{2} \right) U - U \cdot BB \right] = 0,$$

$$\frac{\partial B}{\partial t} + \nabla \cdot (UB - BU) = 0,$$

with the magnetic field subjected to Gauss law:

$$\nabla \cdot B = 0.$$  

This system expresses the conservation of mass, momentum, total energy and magnetic field.
We have chosen units such that the vacuum magnetic permeability ($\mu_0$) is equal to unity. A
global pressure can be defined by the sum of the mechanical and magnetic pressures:

\[ p_G = p + \frac{B \cdot B}{2}. \]  

(6)

In the proposed algorithm, the standard pressure, \( p \), is calculated indirectly via an equation assembled using both the continuity (Eq. 1) and the momentum (Eq. 2) equations, as it is usual in pressure-correction algorithms. Temperature is a derived quantity and is obtained through an equation of state:

\[ T = \frac{1}{c_v} \left[ e - \frac{1}{2} \left\{ U \cdot U + \frac{B \cdot B}{\rho} \right\} \right]. \]  

(7)

3 NUMERICAL METHOD FOR MULTIDIMENSIONAL MHD

In the following subsections we describe the method devised for solving the multidimensional MHD equations of the previous section. Firstly, we explain how the set of conservative fluxes are calculated using a modified version of the AUSM method especially developed for the MHD equations. Secondly we briefly describe the numerical technique used to guarantee that the constraint given by \( \nabla \cdot B = 0 \) is satisfied at all times. Finally, all the steps of the solution algorithm that couples the dependent variables are outlined and discussed in detail.

We make use of the OpenFOAM (Field Operation And Manipulation) package as a developer tool for our new code. The OpenFOAM code is an object oriented numerical simulation toolkit for continuum mechanics, written in C++ language, released by Silicon Graphics International Corp. The main advantage of this tool is that it is open source, which allows the user to modify the source code and take advantage of contributions from a worldwide community.

3.1 Conservative Fluxes

The AUSMPW-MHD [5] technique allows us to calculate the set of conservative fluxes which are then assembled as the MHD system of discretized equations. For three dimensional flow, exhibiting variations along the \( x-, y- \) and \( z- \) directions, the system of equations can be written in the following conservative form:

\[ \frac{\partial \mathcal{H}}{\partial t} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}, \]  

(8)

where \( \mathcal{H} \) is the state vector for the conservative variables and \( F_{x,y,z} \) represents the flux vectors in each direction.

\[
\mathcal{H} = \begin{pmatrix}
\rho \\
\rho U_x \\
\rho U_y \\
\rho U_z \\
\rho e \\
B_x \\
B_y \\
B_z
\end{pmatrix}, \\
F_x = \begin{pmatrix}
\rho U_x \\
\rho U_x^2 + p_G - B_x^2 \\
\rho U_x U_y - B_x B_y \\
\rho U_x U_z - B_x B_z \\
(\rho e + p_G) U_x - B_x (B_x U_x + B_y U_y + B_z U_z) \\
0 \\
B_y U_x - B_x U_y \\
B_z U_x - B_x U_z
\end{pmatrix}.
\]  

(9)

With \( F_y \) and \( F_z \) obtained by properly permuting indices. The global pressure and total energy are given by:

\[ p_G = p + \frac{1}{2} \left( B_x^2 + B_y^2 + B_z^2 \right), \]  

(10)
\[ e = \frac{1}{2} \left( U_x^2 + U_y^2 + U_z^2 \right) + C_v T + \frac{1}{2\rho} \left( B_x^2 + B_y^2 + B_z^2 \right). \]  

(11)

Using the AUSM$^+$ method for gas dynamics we can calculate the flux function at the cell face (index \( f \)) as follows:

\[ F_f = a_f \left( \mathcal{M}_4^+ \phi_L + \mathcal{M}_4^- \phi_R \right) \left( P_5^+ P_L + P_5^- P_R \right), \]

(12)

where \( a_f \) is the common speed of sound, \( \phi = (\rho, \rho U, \rho e)^T \) and \( P = (0, p, 0)^T \). The subscripts \( L \) and \( R \) denote the left and right side face contributions, \( \mathcal{M}_4^\pm \) and \( P_5^\pm \) are Mach number interpolation functions. In a previous paper [14] we have implemented such functions for obtaining cell face values of velocity and pressure, and, with the PISO algorithm we have considered the Euler equations at arbitrary Mach numbers. The purpose here is to devise a similar method of assembling the fluxes for the MHD system of equations.

The AUSM method was developed for gas dynamics using the appropriate characteristic speeds and, because of that, it is not readily applicable to calculate MHD flow. The main issue is that the ideal MHD equations have seven different characteristic speeds, instead of three as in the case of the Euler equations. So, in order to properly scale the Mach number interpolation functions, we have decided to adapt to our method the weighting functions developed by Han et al. [5] which do account for the magnetic field. The scaled \( \mathcal{M}_4^\pm \) are then given by:

\[
\begin{align*}
  M_f = \mathcal{M}_4^+ + \mathcal{M}_4^- \geq 0 : & \quad \tilde{\mathcal{M}}_4^- = \mathcal{M}_4^- \cdot w \cdot (1 + f_R), \\
  & \quad \tilde{\mathcal{M}}_4^+ = \mathcal{M}_4^+ + \mathcal{M}_4^- \cdot [(1 - w) \cdot (1 + f_R) - f_L],  \\
  M_f = \mathcal{M}_4^+ + \mathcal{M}_4^- < 0 : & \quad \tilde{\mathcal{M}}_4^+ = \mathcal{M}_4^+ \cdot w \cdot (1 + f_L), \\
  & \quad \tilde{\mathcal{M}}_4^- = \mathcal{M}_4^- + \mathcal{M}_4^+ \cdot [(1 - w) \cdot (1 + f_L) - f_R],
\end{align*}
\]

(13)

where \( f \) and \( w \) are functions based on the global pressure:

\[
\begin{align*}
  f_{L,R} & = \begin{cases} 
    \left( \frac{p_{G,L} + p_{G,R}}{p_{G,s}} \right)^3 - 1, & p_{G,s} \neq 0, \\
    0, & p_{G,s} = 0 
  \end{cases}, \quad p_{G,s} = p_5^+ p_{G,L} + p_5^- p_{G,R}, \\
  w & = 1 - \min \left( \frac{p_{G,L}}{p_{G,R}}, \frac{p_{G,R}}{p_{G,L}} \right)^3, \\
  p_{G,L} & = p_L + \frac{1}{2} (B_x^2 + B_y^2 + B_z^2)_L, \\
  p_{G,R} & = p_R + \frac{1}{2} (B_x^2 + B_y^2 + B_z^2)_R.
\end{align*}
\]

(15)

(16)

(17)

(18)

In addition, the interface Mach number needs to be calculated with the fast magnetosonic speed, and not the standard sound speed as in the AUSM$^+$ method:

\[ M_{L,R} = \frac{U_{L,R}}{c_f}, \]

(19)
where the common magnetosonic fast speed at the cell face is given by:

\[ c_f = \min (c_{f,L}, c_{f,R}), \]  

\[ c_{f,L} = \left\{ \frac{1}{2} \left[ a_L^2 + \frac{B_L^2}{\rho_L} + \sqrt{\left( a_L^2 + \frac{B_L^2}{\rho_L} \right) - 4 a_L^2 \frac{B_{n,L}^2}{\rho_L}} \right] \right\}^{\frac{1}{2}}, \]  

\[ c_{f,R} = \left\{ \frac{1}{2} \left[ a_R^2 + \frac{B_R^2}{\rho_R} + \sqrt{\left( a_R^2 + \frac{B_R^2}{\rho_R} \right) - 4 a_R^2 \frac{B_{n,R}^2}{\rho_R}} \right] \right\}^{\frac{1}{2}}, \]  

where \( a_{L,R} \) are the left- and right-states of the speed of sound at the cell interface and \( B_n = \hat{S}_f \cdot \mathbf{B} \) is the normal component of the magnetic field.

### 3.2 Treatment of the \( \nabla \cdot \mathbf{B} = 0 \) Constraint

For multidimensional MHD flows special care needs to be taken to ensure that the \( \nabla \cdot \mathbf{B} = 0 \) constraint is satisfied or, at least, that its value is small. It has been demonstrated by Brackbill and Barnes [2] that, even if the solenoidal condition is satisfied at the initial time step, numerical errors related to time and space discretization are magnified, as a result of the following evolution equation for \( \nabla \cdot \mathbf{B} \):

\[ \frac{\partial}{\partial t} \left( \nabla \cdot \mathbf{B} \right) = 0 + O \left( \Delta x^m, \Delta t^n \right), \]  

with \( m, n \geq 1 \). To handle this problem we apply the hyperbolic/parabolic divergence cleaning method suggested by Dedner et al. [4]. Their proposal was to couple the \( \nabla \cdot \mathbf{B} = 0 \) constraint to the evolution equation for \( \mathbf{B} \) by means of a scalar function \( \Psi \), through a gradient term,

\[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{U} \mathbf{B} - \mathbf{B} \mathbf{U}) + \nabla \Psi = 0. \]  

Hence an equation needs to be assembled and solved for this scalar \( \Psi \); in our case we have implemented the following hyperbolic/parabolic equation:

\[ \frac{\partial \Psi}{\partial t} + c_h^2 \nabla \cdot \mathbf{B} = -\frac{c_h^2}{c_d} \Psi. \]  

This equation implies that the errors associated to \( \nabla \cdot \mathbf{B} \) are convected by the \( c_h \) speed and, at the same time, are damped by the dissipation coefficient \( c_d \). The \( c_h \) speed is determined by:

\[ c_h = \frac{CFL}{\Delta t \times \max \left( \frac{1}{\Delta x^m} \right)}, \]  

\[ CFL = \max \left[ \frac{\left( |U_f| + c_f \right) \Delta t}{d} \right], \]  

where \( d \) is the cell size and \( U_f = c_f \left( \mathcal{M}_4^1 + \mathcal{M}_4^1 \right) \) is the cell face velocity. With this approach \( c_h \) is the maximum speed that is compatible with the \( CFL \) number. The dissipation coefficient is given by:

\[ c_d = \sqrt{-\Delta t \frac{c_h^2}{\ln (c_r)}}, \]  

with \( 0 < c_r < 1 \),

and in all calculations we set \( c_r = 0.9 \).
3.3 Implementation of the Numerical Algorithm

The algorithm proposed is based on the PISO method of Issa [6]. In a previous paper [14] we described the first version of the algorithm which was then applied to solve the Euler equations at arbitrary Mach numbers. In the present work we explain the steps required to extend the algorithm for the solution of the MHD equations. A segregated approach is followed consisting of successive prediction and correction steps, with values obtained at a previous time step denoted with *, **, ***.

3.3.1 Prediction Step

In this first step all nodal values are assumed to be known at the previous time level \( n \). The interpolation Mach number functions are calculated at the beginning of each time step. These functions allow us to calculate the following sets of fluxes for the three dimensional MHD equations:

\[
\mathcal{F}_f = c_f \left( \bar{\mathcal{M}}_1^+ \Phi_{L}^n + \bar{\mathcal{M}}_1^- \Phi_{R}^n \right) + \left( \mathcal{P}_S^+ P_{L}^n + \mathcal{P}_S^- P_{R}^n \right) + \frac{1}{2} \left( \Phi_{B,L}^n + \Phi_{B,R}^n \right),
\]

\[
\Phi = \begin{pmatrix} \rho \\ \rho U_x \\ \rho U_y \\ \rho U_z \\ B_x \\ B_y \\ B_z \\ \rho e + p G \end{pmatrix}, \quad P = \begin{pmatrix} 0 \\ S_x p G \\ S_y p G \\ S_z p G \\ -B_x U_x \\ -B_y U_y \\ -B_z U_z \\ -B_f (U \cdot B) \end{pmatrix}, \quad \Phi_B = \begin{pmatrix} 0 \\ -B_x B_f \\ -B_y B_f \\ -B_z B_f \\ S_x \Psi \\ S_y \Psi \\ S_z \Psi \\ c_f^2 B_f \end{pmatrix} \tag{30}
\]

Here, \( B_f = S_x B_x + S_y B_y + S_z B_z \), with \( S_x, S_y \) and \( S_z \) being the cell face area components, and \( B_f = (B_{f,L} + B_{f,R})/2 \); see [5] for more details. Because we are using PISO as the base pressure/velocity algorithm we need to remove the magnetic pressure from the global pressure. In this way we can obtain the face value of the thermodynamic pressure which is required in the pressure gradient term of the momentum equation (see [14]):

\[
p^n_f = \mathcal{P}_S^+ p_{L}^n + \mathcal{P}_S^- p_{R}^n. \tag{31}
\]

The first equation to be solved is an explicit version of the continuity equation, based on the mass flux that was previously assembled with the AUSM-MHD method (Eqs. 29 and 30). The solution of this equation gives us a predicted value of density,

\[
\frac{\partial \rho^n}{\partial t} + \nabla \cdot (m^n_f) = 0, \tag{32}
\]

\[
m^n_f = c_f^n \left( \bar{\mathcal{M}}_1^+ \rho_{L}^n + \bar{\mathcal{M}}_1^- \rho_{R}^n \right). \tag{33}
\]

After this, an explicit equation for each component of the magnetic field is solved,

\[
\frac{\partial B^n_f}{\partial t} + \nabla \cdot (B^n_f) = 0, \tag{34}
\]
\[
B_j^n = c_j^n \left( \mathcal{M}_4 B_{,i,L}^n + \mathcal{M}_4 B_{,i,R}^n \right) - \left( \mathcal{P}_5^+ \left[ (\bar{B}_j)^n U_i^R \right] + \mathcal{P}_5^- \left[ (\bar{B}_j)^n U_i^R \right] \right) \\
+ \frac{1}{2} \left( [S_i \Psi^n]_L + [S_i \Psi^n]_R \right),
\]
where the subscript \(i\) represents the multiple components of vector \(\mathbf{B}\) (\(i = x, y, z\)). For 1D flows, with variations along the \(x\)-axis, in order to satisfy the \(\nabla \cdot \mathbf{B} = 0\) condition we must impose \(B_x = \text{const.}\). To obey this condition we neglect the \(B_x\) equation and just solve for \(B_y\) and \(B_z\). These equations are based on fluxes that were calculated previously with the AUSM-MHD method and allow us to obtain intermediate values of \(B_x^n, B_y^n,\) and \(B_z^n\).

Predicted values of the velocity field, \(\mathbf{U}^*\), at the present time step are obtained by solving an explicit equation for each direction,

\[
\frac{\partial (\rho^* U_i^*)}{\partial t} + \nabla \cdot (\mathbf{U}_j^*) = -\nabla p^*_j,
\]

\[
U_j^* = c_j^n \left( \mathcal{M}_4^+ [\rho^*_L U_i^L] + \mathcal{M}_4^- [\rho^*_R U_i^R] \right) + \left( \mathcal{P}_5^+ \left[ S_i (B^*)^2 \right]_L \right) + \mathcal{P}_5^- \left[ S_i (B^*)^2 \right]_R \\
- \frac{1}{2} \left( [(\bar{B}_j)^n B_i^*]_L + [(\bar{B}_j)^n B_i^*]_R \right).
\]

The pressure gradient and the magnetic field terms are treated in an explicit way using existing magnetic field values \(B_x^n, B_y^n,\) and \(B_z^n\) (Eq. 34), and face values of pressure are calculated with Eq. 31. The discretized momentum equation for \(\mathbf{U}^*\) is given by,

\[
a_p^U \mathbf{U}^* = \mathbf{H} (\mathbf{U}^n) - \nabla p_j^n,
\]

where \(a_p^U\) is the central velocity coefficient and the operator \(\mathbf{H} (\mathbf{U}^n)\) is build using the convective terms of neighbor cells to \(P\), the magnetic explicit terms and the explicit part of the time derivative:

\[
\mathbf{H} (\mathbf{U}^n) = \sum a_N^U \mathbf{U}_N^n + S_R^n + \frac{\mathbf{U}^n}{\Delta t}.
\]

The last equation to be solve before the PISO correction cycle is an equation for the total energy,

\[
\frac{\partial (\rho^* e^*)}{\partial t} + \nabla \cdot (\mathbf{E}_j^n) = 0,
\]

\[
\mathbf{E}_j^n = c_j^n \left( \mathcal{M}_4^+ \left[ \rho^*_L e^*_L + \frac{(B^*)_L^2}{2} + p_L^n \right] + \mathcal{M}_4^- \left[ \rho^*_R e^*_R + \frac{(B^*)_R^2}{2} + p_R^n \right] \right) \\
- \left( \mathcal{P}_5^+ \left[ (\bar{B}_j)^n (\mathbf{U}^* \cdot \mathbf{B}^*) \right]_L \right) + \mathcal{P}_5^- \left[ (\bar{B}_j)^n (\mathbf{U}^* \cdot \mathbf{B}^*) \right]_R,
\]

followed by a temperature, \(T^*\), update using the equation of state,

\[
T^* = \frac{1}{c_v} \left[ e^* - \frac{1}{2} \left( \frac{(U^*)^2}{\rho^*} + \frac{(B^*)^2}{\rho^*} \right) \right].
\]

With this this new temperature, new values of the compressibility coefficient are evaluated, \(\psi^* = 1/RT^*\), and density is updated, \(\rho^{**} = \psi^* \rho^n\).
3.3.2 Correction Step

The $H(U^n)$ operator gives an intermediate velocity field which does not take into account the effect of pressure (see [14] for more details). Mach number interpolation functions are calculated once again inside the PISO cycle, with the AUSM-MHD method. These new functions serve to calculate the sonic flux to be used in the pressure equation,

$$F^*_s = c^*_s \left( \mathcal{M}^+_4 \psi^*_L + \mathcal{M}^-_4 \psi^*_R \right).$$  \hfill (43)

This pressure equation is built and solved using the previously obtained values for compressibility, $\psi^*$, and density, $\rho^{**}$, as:

$$\frac{\partial (\psi^* p^*)}{\partial t} + \nabla \cdot \left( F^*_s p^* \right) - \nabla \cdot \left( \frac{\rho^{**}}{\bar{a}_{\rho}} \nabla p^* \right) = 0,$$  \hfill (44)

and gives the predicted value for the pressure, $p^*$. The velocity field is corrected in an explicit way using the new pressure gradient and the first predicted velocity. The pressure gradient is again calculated with the pressure face value calculated using interpolated Mach number functions,

$$U^{**} = \frac{H(U^n) - \nabla p^*_f}{\bar{a}_{\rho}}.$$  \hfill (45)

Finally, density is corrected, again using the equation of state, $\rho^{***} = \psi^* p^*$. This cycle should be repeated until the continuity equation is satisfied and, in all calculations we have used two correction steps.

4 TEST CASES

In the following subsections we present some of the test cases used for validation. First we calculate the circularly polarized Alfvén waves using a 1D and 2D formulations. With this test case we intend to validate our solver for smooth flows. Then we consider several MHD cases with discontinuities in both velocity and magnetic field. For that purpose the rotated 1D Riemman problem and the Orszag-Tang vortex problem are computed.

4.1 Circularly polarized Alfvén waves

The first set of test cases allows us to validate the accuracy of our method for smooth flows. These test cases are standard problems for MHD schemes [3, 12], and they offer analytical solutions of the MHD equations for arbitrary amplitudes. They can be calculated using a one-dimensional formulation of the MHD equations in a 1D grid. But it is also possible to rotate the geometry at an angle $\alpha$ with respect to the $x-$axis and in this case the problem requires a 2D solution.

The boundary conditions are periodic with $x \in [0;1]$, for the 1D case and with $(x, y) = [0; 1/ \cos \alpha] \times [0; 1/ \sin \alpha]$ for the 2D case. The initial conditions are: $\rho = 1; \gamma = 5/3; U_{\parallel} = 0; B_{\parallel} = 0; U_\perp = B_\perp = 0.1 \sin [2\pi (x \cos \alpha + y \sin \alpha)]; U_z = B_z = 0.1 \cos [2\pi (x \cos \alpha + y \sin \alpha)]$.

The Alfvén speed is $|U_A| = B_\parallel / \sqrt{\rho} = 1$, in this case at $t = 1[s]$ the flow is expected to return to its initial state. The $x$ and $y$ components of the magnetic field are given by: $B_x = B_\parallel \cos \alpha - B_\perp \sin \alpha$; and $B_y = B_\parallel \sin \alpha + B_\perp \cos \alpha$ (similarly for velocity). As expected, for 1D flow $B_x = B_\parallel$ and $B_y = B_\perp$.

In Fig.1 we present the results obtained for the magnetic field components. These are for the 1D case ($\alpha = 0$) with grid resolutions of $N = 128$, $N = 64$, $N = 32$ and $N = 16$, and
the profiles of the $B_y$ and $B_z$ magnetic components are shown after five periods. We see that regarding the wave amplitude even the lower grid resolution results agree well with the initial exact solution, which was obtained with the finer grid. If we compare these same results with those obtained using density based solvers described in literature (see for example the results obtained by Tóth [12]) we can see that, for a low grid resolution, the wave damping is generally quite severe, a feature which is absent in our case. However, our results do present a small phase error which is almost completely removed when the grid resolution is increased.

![Graphs](image.png)

Figure 1: Results obtained for the $B_y$ and $B_z$ components of the magnetic field, calculated with three different grid resolutions. The continuous line shows the initial distribution calculated on the $N = 128$ grid.

The second test is the corresponding 2D case, which is solved on a $N \times N$ grid making an angle $\alpha = 30^\circ$ with the $x-$axis. We have performed this test with three successive grid resolutions: $N = 64$, $N = 32$ and $N = 16$. In Fig.2 we present the results for the perpendicular ($B_\perp = B_y \cos \alpha - B_x \sin \alpha$) and z-components of the magnetic field vector projected against a vector parallel to the direction of the wave propagation $r_\parallel = x \cos \alpha + y \sin \alpha$. We plot the solution obtained using the three grid resolutions after five periods and, for comparison, we also show the initial solution on the finest $N = 64$ grid. This figure demonstrates that, on the finest grid, the magnetic field returns to its initial state, as expected. We can see that the wave amplitude is again maintained. Although more tests are needed these results are a good indicator that the all-Mach pressure-based solver, presented here for the MHD equations, is fulfilling its purpose.

4.2 Rotated shock tube

In this section we discuss the results for the 1D Riemman problem in a two dimensional perspective. Tóth [12] have shown that the 1D shock tube problem can be converted into a 2D case using a $N \times 2$ grid if the discontinuous interface lies at an angle $\alpha$ to the $y-$axis. The computational domain is a narrow strip with $(x, y) = [0, 1] \times [0, 2/N]$ and the rotation angle is $\alpha \approx 63.4^\circ$. For the superior and inferior border we have imposed shifted periodic boundary conditions and for the remaining borders all variables were fixed, see Fig.3. All interpolations required relied on the high resolution CUBISTA scheme [1].
The test case is a 2.5D solution of the rotated 1D Riemann problem with the following initial left and right states,

\[
\begin{pmatrix}
\rho \\
U_\parallel \\
U_\perp \\
U_z \\
p \\
B_\parallel \\
B_\perp \\
B_z \\
\end{pmatrix}_L = \begin{pmatrix}
1.08 \\
1.2 \\
0.01 \\
0.05 \\
0.95 \\
2/\sqrt{4\pi} \\
3.6/\sqrt{4\pi} \\
2/\sqrt{4\pi} \\
\end{pmatrix}, \quad \begin{pmatrix}
\rho \\
U_\parallel \\
U_\perp \\
U_z \\
p \\
B_\parallel \\
B_\perp \\
B_z \\
\end{pmatrix}_R = \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
1 \\
2/\sqrt{4\pi} \\
4/\sqrt{4\pi} \\
2/\sqrt{4\pi} \\
\end{pmatrix}.
\]

For the adiabatic index we have used $\gamma = 5/3$.

In Fig.4 we present results obtained at $t = 0.2 \cos \alpha$. The plots show results for the 2D solution compared with a high resolution 1D solution ($N = 1024$). We can see that the 2D results look somehow spread for a $N = 256$ grid resolution. However, it was noticed by Tóth [12] that using a $N \times 2$ grid approach will give us an effective resolution 3 times lower than that on a $N \times N$ grid for the same value of $N$. Nevertheless, using such a procedure turns out
to be more economic and we can still check that our two dimensional results capture the correct physics of the problem. From left to right we have: fast shock; rotational discontinuity; slow shock; contact discontinuity; slow shock; rotational discontinuity; and fast shock. We note, in part (i) of this figure, that the divergence of $\mathbf{B}$ only exhibits clear non-zero values close to the two distinct discontinuities in the perpendicular component of $\mathbf{B}$, thus supporting the present treatment of that constraint.

### 4.3 Orszag-Tang vortex

The final test case is the Orszag-Tang vortex, which is a well known two-dimensional test case for MHD schemes [15, 3, 12, 8]. Starting from smooth initial conditions this test case leads to a system of complex MHD shocks, making this problem an ideal test for assessing the ability
of the method to handle the complex interaction of several MHD discontinuities included in the evolution of the vortex. The computational domain is a square box with \((x, y) = [0; 2\pi] \times [0; 2\pi]\) and periodic boundary conditions are applied everywhere. The initial conditions are given by:
\[
\rho = \gamma^2; \quad U_x = -\sin y; \quad U_y = \sin x; \quad U_z = 0; \quad B_x = -\sin y; \quad B_y = \sin 2x; \quad B_z = 0; \quad p = \gamma; \quad \gamma = 5/3.
\]

The CUBISTA scheme was, again, used for variable interpolation. The problem was calculated on a \(N \times N\) grid using three different resolutions (\(N = 200\), \(N = 400\) and \(N = 800\)).

In Fig. 5-a), b) and c) we present the predicted results in terms of pressure contour lines calculated at \(t = \pi\) on the three grids. Careful inspection of these plots allows us to infer that the MHD shock interaction is calculated with accuracy for all values of \(N\). Additional more quantitative comparison requires data from other sources. In this case we have chosen to compare our results with those obtained by Miyoshi and Kusano [8]. These authors have adequate data for comparison because they used the same choices for length and time units, and they employed a very efficient and accurate MHD scheme for the ideal MHD equations. Such comparison is presented in Fig. 5-d), showing the pressure distribution along the line \(y = 0.64\pi\).

We see that our method performs well, although some discrepancies are observed in the initial part of this particular pressure profile. Even Miyoshi and Kusano have commented that more dissipative schemes result in this types of discrepancies, which are noticeable in the two longer discontinuities, where the Roe scheme use by the authors seems to present sharper shock wave resolution.

Figure 5: Numerical results for the Orszag-Tang vortex at \(t = \pi\). Pressure contour lines calculated on grids with: a) \(N = 200\); b) \(N = 400\); c) \(N = 800\). d) Comparison between current predictions and results by Miyoshi and Kusano [8] with a Roe type solver, for the section a-a defined in a).
5 CONCLUSION

In the current report we have presented preliminary results of a new method for solving the ideal MHD equations. The test results for the Alfvén waves show a small phase error. However, we have proved that this new algorithm is more accurate in what regards calculation of the wave amplitude when compared to density solvers currently found in the literature.

On the other hand, the results for the shock tube problem have clearly demonstrated that a pressure based algorithm can calculate highly compressible MHD flow, including the complex discontinuities that are associated to this hyperbolic system of equations. The Orszag-Tang vortex problem have shown that the new formulation was able to capture the complex physics of shock wave interaction found in some MHD problems. Further, the method proposed is shown to be competitive when compared with other solvers, which have proved to be highly accurate in the calculation of compressible MHD flows.

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REFERENCES


